A generalised framework for defining discrete Wigner functions via the Gottesman-Kitaev-Preskill code

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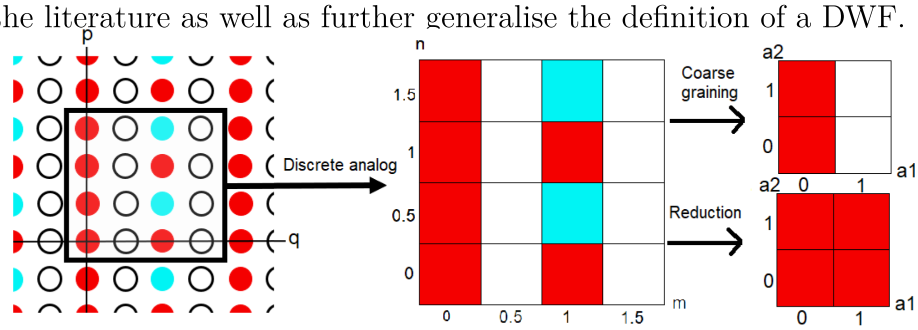
Wigner functions (WFs) are a useful tool in quantum mechanics to represent states and operators in phase space. Additionally, for states, the negativity of the WF can be a resource for quantum computing [1]. However, unlike the uniquely defined continuous WF (CWF), for discrete systems, there are many definitions of a discrete WF (DWF), each defined over some subset of dimensions (even, odd, or prime and prime power) [1–3], and sizes of phase space.

In this work, we construct a general framework for deriving DWFs based on a $d \times d$ sized phase space (a lattice with d^2 number of points) from a DWF defined over a doubled (a lattice of size $2d \times 2d$ and so $4d^2$ number of points) phase space. We construct the doubled DWF by focusing on a $2d \times 2d$ sized unit cell for the CWF of a Gottesman-Kitaev-Preskill (GKP) [4, 5] encoded state, where such a DWF(i.e., a doubled one) contains redundant information. This doubled DWF is the same as what Leonhardt [6], Feng et al [5], and Hannay et al [7] have defined.

To remove this redundancy and construct a distinct $d \times d$ DWF, we use cross-correlations between the doubled DWF and a choice of stencil $M \in \mathcal{M}$ ($\mathcal{M}$ is a special class of maps) where these Ms average the information present in the doubled phase space in some manner and before the phase space is halved. we call these M-DWFs. By identifying a subset of maps, we show that our general framework reproduces the DWFs defined by Wootters' [2], Gross [8], Leonhardt [6], and Cohendet [3]. Additionally, and interestingly, we prove that one of our maps called the Coarse grain map (CGM) produces a working DWF for all even dimensions and is based on a $d \times d$ sized lattice, i.e., with no need to double the phase space size.

Additionally, We further find that these different DWFs can be straightforwardly related to one another through the use of a super operator framework in such a way that the properties from one DWF definition can be related to analogous properties in a different definition.

By providing a straightforward procedure from the CWF to a DWF through the GKP code and creating a general framework, we begin to establish common ground amongst the plethora of DWFs in the literature as well as further generalise the definition of a DWF.



Left: CWF of a $d = 2$ logical-0 GKP state with a unit cell indicated. White spaces are zero-valued, while red and blue are equivalent positive and negative values. Middle: The associated $2d \times 2d$ DWF. Right: Coarse grain mapped DWF and reduction mapped DWF.

References

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