Improved Separators by Multigrid Methods

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Abstract: The graph partitioning problem has many different applications. One of them is the partitioning problem in parallel computations.
With respect to process and device simulation we see two direct connections:
– for some parallel solution methods we are interested in device simulation, good approximations of best separators are essential;
– the equations to be solved here have properties pretty close to those of diffusion–convection equations – so the problem is a test case for the algebraic MG algorithm aiming on the device equations.
The combinatorial problem is known to be NP hard – so different types of heuristic solutions are in use: stimulated annealing at the one end and the approximate solution of an analytic analog – the Neumann eigenvalue problem – at the other. By means of multigrid algorithms wider classes of problems, not only Neumann eigenvalue problems, can and will be efficiently solved. Therefore we are interested in more general analytic analogs, demonstrate the possibility to solve the related discrete problems and the potential of improvement.
The original problem has the form: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph, with the set of vertices $\mathcal{V}$, edges $\mathcal{E}$ and numbers of elements $v = |\mathcal{V}|$, $e = |\mathcal{E}|$.
One asks for two subsets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with
a) the number of edges connecting $\mathcal{V}_1$. $\mathcal{V}_2$ minimal, and
b) $|v_1 - v_2| \to \min$.

The simplest analytical analog is a Neumann eigenvalue problem (well known in the literature) – but it can be generalized to the following one:

$$\tilde{\mathcal{L}}_\alpha(u, \lambda) := \frac{|||\nabla u|||^2}{\|h(u)\|^2} + \lambda_2 \frac{f_\alpha^2(u)}{\|h(u)\|^2} \to \min . \qquad (1)$$

Here $|||\nabla u|||^2/\|h(u)\|^2$ is the Lagrange function, $|||\nabla u|||^2$ is a suitable norm of the gradient in analogy to a) and $f_\alpha(u)$ represents an approximation of the side condition b) replaced by

$$f(u) = \int_{\text{supp } u^+} dV - \int_{\text{supp } u^-} dV = 0, \qquad (2)$$

where supp u^+, supp u^- means the support of the positive / negative part of u.
Using the L_2 norm of the gradient, $h(s) = s$ and relaxing (2) to

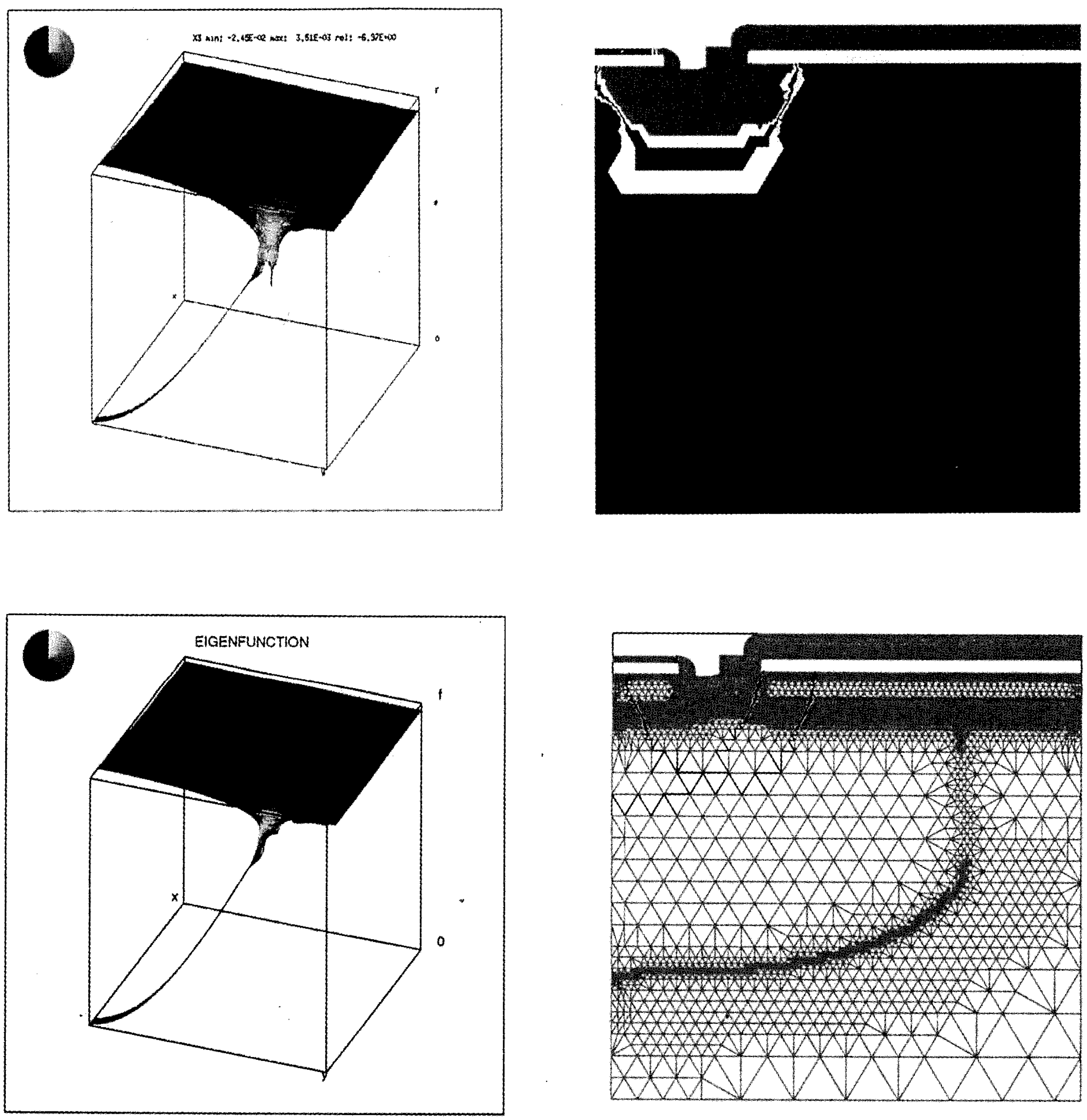
$$\tilde{f}(u) = \int u dV = 0$$

yields the Neumann eigenvalue problem. In this case b) has to be approximated afterwards. This is done by an ordering and counting procedure or finding a constant c that fulfills

$$\int_{\text{supp }(u+c)^+} dV - \int_{\text{supp }(u+c)^-} dV = 0.$$

Approximating the L_1 norm of the gradient and the signum function by f_α, $\alpha \to 1$, yields a family of problems requiring the solution of symmetric indefinite problems, some of which exhibiting strong variations in the coefficients.

This makes the problem attractive for testing an algebraic MG algorithm. Still at the beginning of the exploration of the many different possibilities, some examples (non-symmetric ones – the Neumann eigenvalue problem is a symmetry indicator) show potential for improvements compared with the Neumann eigenvalue problem solutions.



Figures: the solution for $\alpha = 0.60634$ and the first eigenfunction of Neumann eigenvalue problem (left).
Separator for $\alpha = 0.75395$ and comparison of the separator due to the Neumann eigenvalue problem (black, 143 cut edges) and the best one (blue, 80 cut edges) known until now (right).